(see, e.g. [1]). In these cases, the original equation has a non-stationary distribution with density $f(z-ct)$ (a "soliton" distribution). Note that such systems arise in the dynamics of variable-mass stochastic systems.

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THE STABILITY OF A CLASS OF REVERSIBLE SYSTEMS†

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The problem of the stability of the point of rest of an autonomous system of ordinary differential equations from a class of reversible systems [1] characterized by the critical case of m zero roots and n pairs of pure imaginary roots is considered. When there are no internal resonances [2, 3], the point of rest always has Birkhoff complete stability [2]. Internal resonances may lead to Lyapunov instability. The conditions of stability and instability of the model system when there are third-order resonances may be obtained from a criterion previously developed [4] for the case of pure imaginary roots. The results are used to analyse the stability of the translational–rotational motion of an active artificial satellite in a non-Keplerian circular orbit, including a geostationary satellite in any latitude [4, 5]. The region of stability of relative equilibria and regular precession of the satellite is constructed assuming a central gravitational field and the resonance modes are analysed.

1. CONSIDER the system of equations of perturbed motion

$$\mathbf{X}^{\cdot} = D\mathbf{X} + \Phi(\mathbf{X});\quad \mathbf{X} \in R^N;\quad \Phi(0) = 0 \tag{1.1}$$

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where D is a constant square matrix and $\Phi(\mathbf{X})$ is a holonomic vector function whose expansion in powers of the perturbations starts with terms of not lower than the second order.

Assume that system (1.1) is reversible in the sense of [1], i.e. there exists a linear automorphism of the form [6]

$$\mathbf{X} \to Q\mathbf{X}, \quad t \to -t, \quad Q^2 = E \tag{1.2}$$

where Q is a non-singular square matrix and E is the identity matrix. Then there exists a linear transformation to new variables $\mathbf{U}$, $\mathbf{V}$ which diagonalizes the matrix D and the automorphism (1.2) takes the form

$$\mathbf{U} \to \mathbf{U}, \quad \mathbf{V} \to -\mathbf{V}, \quad t \to -t \tag{1.3}$$

The dimensions of the vector $\mathbf{U}$ are clearly the number of $+1$ eigenvalues of the matrix Q and the dimensions of the vector $\mathbf{V}$ is the number of -1 eigenvalues. From (1.2) it follows that all the eigenvalues of Q are ± 1. Assume additionally that $\dim \mathbf{U} > \dim \mathbf{V}$. Then in the new variables system (1.1) takes the form

$$\begin{gathered}\mathbf{U}^{\cdot} = A\mathbf{V} + F_u(\mathbf{U}, \mathbf{V}), \\ \mathbf{V}^{\cdot} = B\mathbf{U} + F_v(\mathbf{U}, \mathbf{V}), \quad \mathbf{U} \in R^{m+n}, \quad \mathbf{V} \in R^n, \quad m + 2n = N \\ (F_u(\mathbf{U}, -\mathbf{V}) = -F_u(\mathbf{U}, \mathbf{V}), \quad F_v(\mathbf{U}, -\mathbf{V}) = F_v(\mathbf{U}, \mathbf{V}))\end{gathered} \tag{1.4}$$

where A and B are constant rectangular matrices and F_u and F_v are functions analogous to Φ in (1.1).

Analysing the structure of the matrix of the first-approximation system for (1.1), we can show that its characteristic equation has no fewer than m zero roots with m groups of solutions. The remaining $2n$ roots may be treated [1] (in the case of first-approximation stability and when there is no supplementary singularity) as pure imaginary and different.

It follows from the above that we can transfer from the variable $\mathbf{U}$, $\mathbf{V}$ to new variables $\xi \in R^m$, $\zeta \in C^n$, $\bar{\zeta} \in C^n$ in which the matrix D is diagonal. The required transformation matrix has the following block structure:

$$P = \begin{Vmatrix} M & F & F \\ 0 & iS & -iS \end{Vmatrix}$$

Here M, F and S are certain $(m+n) \times m$, $(m+n) \times n$, and $n \times n$ real rectangular matrices, respectively.

The inverse transformation matrix has the form

$$P^{-1} = \begin{Vmatrix} T & 0 \\ K & iL \\ K & -iL \end{Vmatrix}$$

where T, K, L are also some $m \times (m+n)$, $n \times (m+n)$, and $n \times n$ real matrices, respectively.

From the structures of the matrix P^{-1} and the existence of the automorphism (1.3) for system (1.4) it follows that in the new variables the system has the automorphism

$$\xi \to \xi, \quad \zeta \to \bar{\zeta}, \quad \bar{\zeta} \to \zeta, \quad t \to -t \tag{1.5}$$

It can be shown that as a result the power series on the right-hand sides of the equations have purely imaginary coefficients; the same also holds (as in the case with only purely imaginary roots [1, 7]) for the system obtained after non-linear normalization [2, 3]. Therefore, when there are no internal resonances, we obtain after non-linear normalization of (1.1)

$$\begin{aligned} z_s^{\cdot} &= z_s f_s(x_1, \ldots, x_m, z_1\bar{z}_1, \ldots, z_n\bar{z}_n) \\ \bar{z}_s^{\cdot} &= \bar{z}_s \bar{f}_s(x_1, \ldots, x_m, z_1\bar{z}_1, \ldots, z_n\bar{z}_n) \\ x_k^{\cdot} &= 0, \quad (k = 1, \ldots, m; \ s = 1, \ldots, n) \end{aligned} \tag{1.6}$$

where f_s and f_s are formal series with purely imaginary coefficients. As we know [2], the trivial solution of system (1.6) (and therefore of the original system) is Birkhoff completely stable and if the normalizing transformation converges it is also Lyapunov stable.

2. Let us now consider the resonance case, when the stability is determined by the first non-linear terms. The roots $i\lambda_s$ must satisfy the relationship (for third-order resonance [2, 3])

$$\langle p, \lambda \rangle = 0, p_1 + \ldots + p_l = 3, 2 \leqslant l \leqslant 3, \lambda = (\lambda_1, \ldots, \lambda_l) \tag{2.1}$$

where $p = (p_1, \ldots, p_l)$ is an integer vector with prime non-negative components. Then, changing in the normal form to polar coordinates r_β, θ_β by the formulas $z_\beta = r_\beta^{1/2} e^{i\Theta\beta}$, $\bar{z}_\beta = r_\beta^{1/2} e^{i\Theta\beta}$, we obtain the equations

$$\begin{gathered} \dot{x}_k = O(|x|^3, |x||r|, |r|^2) \quad (k = 1, \ldots, m) \\ r_s^{\cdot} = b_s \sin\theta \prod_{j=1}^{l} r_j^{p_j/2} + O(|x|^3, |r|^{3/2}) r_s^{1/2}, \quad (s = 1, \ldots, l) \\ \theta^{\cdot} = \sum_{s=1}^{l} \frac{1}{r_s} \Big[\sum_{k=1}^{m} p_s c_{sk} x_k + p_s b_s \cos\theta \prod_{j=1}^{l} r_j^{p_j - \delta_{sj}} + O(|x|^3, |r|^{3/2}) \Big] \\ \theta = p_1\theta_1 + \ldots + p_l\theta_l, \quad x = (x_1, \ldots, x_m), \quad r = (r_1, \ldots, r_n) \end{gathered} \tag{2.2}$$

Here b_s and c_{sk} are some real constants and δ_{sj} is the Kronecker delta.

If we set $x = 0$ in the model system obtained from (2.2) by omitting the tail of the expansion, this system takes the same form as in the case with only purely imaginary eigenvalues [3, 8]. This means that if the model system was unstable without zero roots, it will also remain unstable with zero roots. On the other hand, if $x \neq 0$, we can construct a sign-definite integral if it exists for $x = 0$. This combined with known stability results for third-order resonance suggests the following theorem.

Theorem 1. If all the constants b_s in Eqs (2.2) are non-zero and there is at least one pair b_μ, b_ν such that $b_\mu b_\nu < 0$, then the point of rest of the model system corresponding to (2.2) is Lyapunov stable. Otherwise, it is unstable.

Stability can be proved using the integral

$$F = \frac{1}{2} \sum_{k=1}^{m} x_k^2 + \sum_{s=1}^{l} \gamma_s r_s + \sum_{\alpha=l+1}^{n} r_\alpha$$

which is sign-definite ($\gamma_s > 0$) when the stability condition of Theorem 1 is satisfied. The instability follows from the existence of the increasing solution

$$\begin{gathered} x_k^* = 0, \ k = 1, \ldots, m; \ r_\alpha^* = 0, \ \alpha = l + 1, \ldots, m \\ \theta^* = \pm \frac{\pi}{2}, \quad r_s^* = \frac{|b_s|}{(a - t)^2} \Big(\prod_{j=1}^{l} |b_j|^{p_j} \Big)^{-1}, \quad s = 1, \ldots, l \end{gathered} \tag{2.3}$$

This instability implies instability of the full system, i.e. we have the following theorem.

Theorem 2. If the model system corresponding to (2.2) has a solution (2.3), then the trivial solution of system (2.2) is unstable.

The proof relies on the considerations of [9] (a different proof apparently can be based on [10]). To this end, we change in (2.2) to the coordinates θ, ρ, $y_l, \ldots, y_m, \varphi_1, \ldots, \varphi_{l-1}, r_{l+1}, \ldots, r_n$ by the formulas

$$r_s = \sigma_s \rho \cos \varphi_s \prod_{j=1}^{s-1} \sin \varphi_j, \quad s = 1, \ldots, l-1$$
$$r_l = \sigma_l \rho \prod_{j=1}^{l-1} \sin \varphi_j \tag{2.4}$$
$$\rho y_k = x_k, \quad k = 1, \ldots, m; \quad \sigma_\beta = |b_\beta| \prod_{j=1}^{l} |b_j|^{p_j/2}, \quad \beta = 1, \ldots, l$$

and introduce the perturbations $\eta = \theta - \theta^*$, $\xi_s = \varphi_s - \varphi_s^*$, $s = 1, \ldots, l-1$, where φ_s^* are the values of φ_s on the solution (2.3).

The equations for the new variables are written as

$$y_k^{\cdot} = -2l^{-1/4}\rho^{1/2} y_k [1 + O(|\xi|)] + O(\rho^3 |y|^3, \rho |y| |\rho_*|, |\rho_*|^2), \; k = 1, \ldots, m$$
$$\rho^{\cdot} = 2l^{-1/4}\rho^{3/2}[1 + O(|\xi|, \rho^2 |y|^3, |P^{1/2}|^3 \rho^{-1})]$$
$$\eta^{\cdot} = -3l^{-1/4}\rho^{1/2} + \rho \sum_{s=1}^{l} \sum_{k=1}^{m} p_s c_{sk} y_k + \rho^{1/2} O(|P^{1/2}|^3 \rho^{-1}, |\xi|^2) \tag{2.5}$$
$$\xi_s^{\cdot} = -2l^{-1/4}\rho^{1/2}\xi_s + O(|P^{1/2}|^3, |\xi|^2 \rho)\rho^{-1/2}, \quad s = 1, \ldots, l-1$$
$$r_\alpha^{\cdot} = O(|P^{1/2}|^3) r_\alpha^{1/2}, \quad \alpha = l+1, \ldots, n$$

Here

$$\xi = (\xi_1, \ldots, \xi_{l-1}, \eta), \; y = (y_1, \ldots, y_m), \; \rho_* = (\rho, r_{l+1}, \ldots, r_n)$$
$$P^{1/2} = (\rho y_1, \ldots, \rho y_m, \rho^{1/2}, r_{l+1}^{1/2}, \ldots, r_n^{1/2})$$

Consider the function [11]

$$2V = \gamma^2 \Big(\rho - \sum_{\alpha=l+1}^{n} r_\alpha\Big) - \sum_{k=1}^{m} y_k^2 - \eta^2 - \sum_{s=1}^{l-1} \xi_s^2$$

which together with its derivative $V^{\cdot}$ satisfies, in view of (2.5), the condition $VV^{\cdot} > 0$ in the region $V > 0$ for sufficiently large γ and is therefore a Chetayev function for system (2.5).

Remark. 1. The change of x_k to y_k in (2.4) does not prevent the application of Chetayev's instability theorem, because we only consider motion in the region $\rho > |y|^2/\gamma^2$.

2. The result is also valid when the entire system (1.1) is reversible, and not only the model part obtained from (1.1) by omitting all terms of higher than second order.

3. Let us apply the results to a stability analysis of the translational–rotational motion of an active satellite, which is maintained by a weak jet-propulsion acceleration in a circular orbit whose plane does not pass through the Earth's centre. This motion is of interest, in particular, for geostationary satellites maintained on an arbitrary latitude [4, 5]. We will assume that the acceleration produced by the jet engines is constant in the attached coordinate system $CX_1Y_1Z_1$ with the axes aligned along the principal central axes of inertia of the satellite; the acceleration vector **w** passes through the satellite's centre of mass C. The satellite is regarded as a rigid body of variable mass which preserves the similarity of the ellipsoid of inertia as the mass is consumed.

The presence of constant jet-propulsion acceleration makes is possible to maintain the centre of mass in a motion such that

$$R = R_0\,(\text{const}),\ \varphi = \varphi_0\,(\text{const}),\ \theta_1 = \omega_1 t + \text{const}$$

where R, φ and θ_1 are respectively the radius, latitude and longitude of the satellite's centre of mass in a spherical geocentric system of coordinates rotating with the Earth. This motion of the satellite is possible in two cases. In the first case (relative equilibrium), the satellite has a triaxial ellipsoid of inertia and is at rest in unperturbed motion relative to the orbital system $CYXZ$ (its axes are aligned with the unit vectors of the spherical system of coordinates at the point C). In the second case (regular precession), the satellite has a dynamic axis of symmetry pointing in the direction of the acceleration vector **w** and spins uniformly around this axis. The dynamic axis of symmetry is at rest in unperturbed motion in the system $CXYZ$.

Stability analysis of these stationary motions is not simple because the presence of the acceleration vector makes it impossible to separate the motion of the centre of mass from the rotational motion and it moreover produces positional forces (in addition to potential and gyroscopic forces), which render the system non-Hamiltonian.

The problem of the stability of relative equilibrium has been considered in [4] with the Earth's gravitational field approximated by the field of a homogeneous triaxial ellipsoid. As a result, the characteristic equation of the system in variations had no zero roots. Allowance for the non-centrality of the field in turn restricted the analysis to geostationary motions in which the centre of mass coincided with one of the points of libration (the points of libration are located in two perpendicular planes, each containing one of the axes of the equatorial section of the Earth's ellipsoid). The results of Secs 1 and 2 enable us to relax this restriction by passing to a central model of the gravitational field. This approach is justified because for geostationary satellites the effect of non-centrality is comparable with lunar–solar perturbations.

To obtain the equations of motion in a central field, it suffices to take $I = \varepsilon = 0$ for the non-centrality parameters in the original system of equations in [4]. In this way, the equation in the longitude θ_1 separates from the main system. The remaining system of equations of 11th order has a family of partial solutions that correspond to the relative equilibrium of the satellite in a circular orbit on a given latitude φ_0:

$$\begin{gathered} R = R_0,\ \varphi = \varphi_0,\ \theta_1^{\boldsymbol{\cdot}} = \omega_1,\ \beta_1 = \gamma_1 = 0,\ \gamma_2 = \sin\lambda \\ p = 0,\ q = \omega_0\cos(\varphi_0 - \lambda),\ r = \omega_0 \sin(\varphi_0 - \lambda) \end{gathered}$$

Here ω_1 is the angular velocity of the satellite's centre of mass relative to the Earth, projected on to the Earth's spin axis, $\omega_0 = \omega_1 + \omega_3$ is the projection of the absolute angular velocity of the centre of mass (ω_3 is the angular velocity of the Earth's rotation), p, q and r are the projections of the satellite's absolute angular velocity on the attached axes and β_1, γ_1 and γ_2 are the direction cosines that specify the relative position of the orbital and the attached coordinate systems.

Introducing the perturbations

$$x_1 = (R - R_0)/R_0,\ x_2 = \varphi - \varphi_0,\ x_3 = (\theta^{\cdot} - \omega_0)/\omega_0,\ x_5 = \varphi^{\cdot}/\omega_0$$
$$x_6 = p/\omega_0,\ x_7 = q/\omega_0 - \cos(\varphi_0 - \lambda),\ x_8 = r/\omega_0 - \sin(\varphi_0 - \lambda)$$
$$x_9 = \beta_1,\ x_{10} = \gamma_1,\ x_{11} = \gamma_2 - \sin\lambda\ (\theta = \theta_1 + \omega_3 t) \tag{3.1}$$

and the dimensionless time $\tau = \omega_0 t$, we can show that the system of equations of perturbed motion admits of the linear automorphism

$$\begin{gathered} x_1 \to x_1,\ x_2 \to x_2,\ x_3 \to x_3,\ x_4 \to -x_4 \\ x_5 \to -x_5,\ x_6 \to -x_6,\ x_7 \to x_7,\ x_8 \to x_8 \\ x_9 \to -x_9,\ x_{10} \to -x_{10},\ x_{11} \to x_{11},\ \tau \to -\tau \end{gathered} \tag{3.2}$$

The existence of this automorphism establishes the reversibility of this system and we obtain the previous case [to prove this, set $I = \varepsilon = 0$ in the equations of perturbed motion [4] and check (3.2) by direct substitution].

From (3.2) and the results of Sec. 1, it follows that in the case of stability to a first approximation the characteristic equation has only purely imaginary and zero roots [it follows from the structure of (3.2) that at least one zero root exists].

Computer calculations have shown that the zero root is unique inside the first-approximation stability region. Thus, further analysis reduces to constructing regions in the parameter space in which none of the roots of the characteristic equation has a real part (as we have noted above, these are regions of Birkhoff complete stability with the exception of resonance sets) and identifying unstable resonance modes. Computer calculations show that the regions obtained by this method are virtually identical (with an accuracy proportional to the parameter $\varepsilon \approx 10^{-6}$) with the regions constructed for a non-central field [4].

Let us now consider the stability of regular precession. In this case, the relative position of the orbital system $CXYZ$ and the attached system $CX_1Y_1Z_1$ is described by Euler angles φ, ϑ and Φ, and not by direction cosines. In the equations of motion of the centre of mass, the derivative of the longitude $\theta_1^{\cdot}$ is conveniently replaced by the projection of the absolute angular velocity of the satellite's centre of mass on to the Earth's spin axis $\omega = \theta_1^{\cdot} + \omega_3$. Then the equations of motion of the centre of mass take the form

$$\begin{gathered} d^2R/dt^2 - R(\omega^2\cos^2\varphi + \varphi^{\cdot 2}) = w\cos\vartheta - \mu/R^2 \\ d(R^2\varphi^{\cdot})/dt + R^2\omega^2\sin 2\varphi/2 = -Rw\cos\psi\sin\vartheta \\ d(R^2\omega\cos^2\varphi)/dt = Rw\cos\varphi\sin\psi\sin\vartheta \end{gathered} \tag{3.3}$$

Here we assume that the acceleration vector $\mathbf{w}$ points along the CZ_1 axis of the attached system of coordinates, which is the satellite's dynamic axis of symmetry; μ is the gravitational parameter of the central field.

To obtain equations that describe the rotational motion of the satellite about the centre of mass, we introduce another (semiattached) system of coordinates $CX'Y'Z'$, in which the axis CZ' coincides with the axis CZ_1, the axis CX' is directed along the line of nodes formed by the planes X_1CY_1 and XCY, and the axis CY' completes the system to a right-hand system. As the phase coordinates we take the Euler angles ψ and ϑ and the projections p, q and r of the satellite's angular velocity vector on the axes of the semiattached system of coordinates.

Without going in detail into complex calculations, we will give the final equations:

$$p^{\cdot} = -\bar{c}qr + q^2 \operatorname{ctg} \vartheta - q\varphi^{\cdot} \sin\psi/\sin\vartheta - q\omega\cos\varphi\cos\psi/\sin\vartheta + 3\mu(\bar{C} - 1)R^{-3}\cos\vartheta\sin\vartheta,\quad q^{\cdot} = \bar{C}pr - pq\operatorname{ctg}\vartheta + p\varphi^{\cdot}\sin\psi/\sin\vartheta + p\omega\cos\varphi\cos\psi/\sin\vartheta,\quad r^{\cdot} = 0$$

$$\psi^{\cdot} = q/\sin\vartheta - \varphi^{\cdot}\sin\psi\operatorname{ctg}\vartheta - \omega(\cos\varphi\cos\psi\operatorname{ctg}\vartheta + \sin\varphi)$$
$$\vartheta^{\cdot} = p + \varphi^{\cdot}\cos\psi - \omega\cos\varphi,\ \sin\psi \tag{3.4}$$

Here $\bar{C}$ is the ratio of the satellite's moments of inertia, and the equation for $\Phi^{\cdot}$ is separated and is not used in what follows.

The system of equations (3.3) and (3.4) has a partial solution that corresponds to regular precession:

$$R = R_0\ (\text{const}),\ \varphi = \varphi_0\ (\text{const}),\ R^{\cdot} = \varphi^{\cdot} = 0,\ \omega = \omega_0\ (\text{const}),\ p = 0$$
$$q = \omega_0\cos(\varphi_0 - \vartheta_0),\ r = \omega_0\sin(\varphi_0 - \vartheta_0) + \omega^*,\ \psi = 0,\ \vartheta = \vartheta_0\ (\text{const})$$

This requires the following relationships between the parameters of the unperturbed motion:

$$\operatorname{tg}\vartheta_0 = \frac{\rho\sin 2\varphi_0}{2(\rho\cos^2\varphi_0 - 1)}\quad \left(\rho = \frac{R_0^3\omega_0^3}{\mu}\right)$$
$$\nu = \frac{\omega_0}{\omega_1^*} = \frac{\bar{C} - 1}{2\bar{C}\cos(\varphi_0 - \vartheta_0)}\left[\frac{3}{\rho}\sin 2\vartheta_0 - \sin 2(\varphi_0 - \vartheta_0)\right]$$

These relationships suggest the existence of a two-parameter family of solutions in the case of regular precession (with the parameters φ_0 and ρ, say). Another relationship determines the quantity ω_0:

$$\left[\frac{\mu}{R_0^4} + \left(\omega_0^2 R_0 - \frac{2\mu}{R_0^2}\right)R_0\omega_0^2\cos^2\varphi_0\right]^{1/2} = w$$

By Malkin's theorem [11], the characteristic equation of the system in variations for this two-parameter family necessarily has two non-zero roots.

Introducing, as in (3.1), dimensionless perturbations and time

$$x_1 = R/R_0 - 1,\ x_2 = \varphi - \varphi_0,\ x_3 = \omega/\omega_0 - 1,\ x_4 = R^{\cdot}/(\omega_0 R_0)$$
$$x_5 = \varphi^{\cdot}/\omega_0,\ x_6 = p/\omega_0,\ x_7 = q/\omega_0 - \cos(\varphi_0 - \vartheta_0)$$
$$x_8 = r/\omega_0 - \sin(\varphi_0 - \vartheta_0) - \nu,\ x_9 = \psi,\ x_{10} = \vartheta - \vartheta_0$$
$$\tau = \omega_0 t$$

we can rewrite the equations of perturbed motion in the form

$$\frac{dx_1}{d\tau} = x_4,\quad \frac{dx_2}{d\tau} = x_5,\quad \frac{dx_3}{d\tau} = 2(1 + x_3)x_5\operatorname{tg}(\varphi_0 + x_2) - 2\frac{x_4(1 + x_3)}{1 + x_1} - \frac{S_1\sin 2\varphi_0\sin x_9}{2(1 + x_1)\sin\vartheta_0\cos(\varphi_0 + x_2)}$$

$$\frac{dx_4}{d\tau} = (1 + x_1)[(1 + x_3)^2\cos^2(\varphi_0 + x_2) + x_5^2] - \frac{1}{\rho(1 + x_1)^2} - \frac{C_1\sin 2\varphi_0}{2\sin\vartheta_0},$$

$$\frac{dx_5}{d\tau} = -2\frac{x_4x_5}{1 + x_1} - \frac{(1 + x_3)^2\sin 2(\varphi_0 + x_2)}{2} + \frac{S_1\sin 2\varphi_0\cos x_9}{2(1 + x_1)\sin\vartheta_0}$$

$$\frac{dx_6}{d\tau} = (\bar{C}_0 + x_7)\left[(\bar{C}_0 + x_7)\frac{C_1}{S_1} - \bar{C}(S_0 + x_8 + \nu) - \frac{x_5\sin x_9}{S_1} - \frac{(1 + x_3)\cos(\varphi_0 + x_2)\cos x_9}{S_1} + \frac{3(\bar{C} - 1)S_1C_1}{\rho(1 + x_1)^3}\right.$$

$$\frac{dx_7}{d\tau} = x_6\left[\bar{C}(S_0 + x_8 + \nu) - (\bar{C}_0^- + x_7)\frac{C_1}{S_1} + (1 + x_3)\ \frac{\cos(\varphi_0 + x_2)\cos x_9}{S_1}\right.$$
$$\frac{dx_8}{d\tau} = 0, \quad \frac{dx_9}{d\tau} = \frac{C_0^+ + x_7}{S_1} - x_5\frac{C_1}{S_1}\sin x_9 - (1 + x_3)\frac{C_1}{S_1}\cos(\varphi_0 +$$
$$+ x_2)\cos x_9 - (1 + x_3)\sin(\varphi_0 + x_2), \quad \frac{dx_{10}}{d\tau} = x_6 + x_5\cos x_9 -$$
$$-(1 + x_3)\cos(\varphi_0 + x_2)\sin x_9$$
$$(C_0^{\pm} = \cos(\varphi_0 \pm v_0), \ S_0 = \sin(\varphi_0 - \vartheta_0), \ C_1 = \cos(\vartheta_0 + x_{10})$$
$$S_1 = \sin(\vartheta_0 + x_{10})).$$

This system admits of a linear automorphism, which, unlike (3.2), does not contain $x_{11} \to x_{11}$ and uses $x_{10} \to x_{10}$ instead of $x_{10} \to -x_{10}$. It is therefore a reversible system of the class considered above. The problem of the stability of regular precession thus reduces to constructing a region of the parameter space in which the roots of the characteristic equation do not have real parts and analysing the internal resonances. Because of the high order of the system, the construction of the stability region in the φ_0, $\bar{C}$ plane and the analysis of third-order internal resonances (as the most dangerous) was carried out numerically by computer. The acceleration **w** was chosen from the condition for it to be a minimum, which necessitates satisfying the relationship [4]

$$\rho_{\text{opt}} = {}^1/_2\,(\sqrt{1 + 8\sec^2\varphi_0} - 1)$$

The hatching in Fig. 1 is the boundary of the Birkhoff complete stability region and the curves crossing the boundary correspond to third-order internal resonances. The solid curves are the resonance sets that cause Lyapunov instability and the dashed curves are the loci where stability is preserved also in the second order. Calculations show that on some curves some of the normal coefficients b_3 vanish. When there are no zero roots, this causes either preservation of stability also in the second order or non-robust instability [8]. These results remain true in our case also, as we see

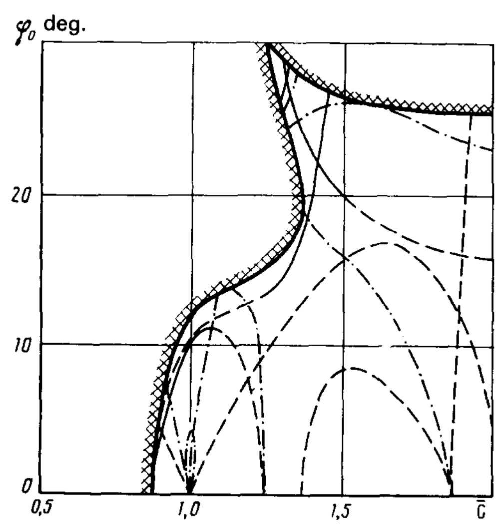

FIG. 1.

from the previous discussion. Resonance sets with non-robust instability are denoted by the dash–dot curve.

Comparing our results with those of [4], we note that the region of stability of regular precession is larger than that of relative equilibrium.

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